A State-Dependent Updating Period For Certified Real-Time Model Predictive Control

Mazen Alamir

***Abstract*—In this paper, a state-dependent control updating period framework is proposed that leads to real-time implementable Model Predictive Control with certified practical stability results and constraints satisfaction. The scheme is illustrated and validated using new certification bound that is derived in the case where the Fast Gradient iteration is used through a penalty method to solve generally constrained convex optimization problems. Both the certification bound computation and its use in the state-dependent updating period framework are illustrated in the particular case of linear MPC. An illustrative example involving a chain of four integrators is used to show the explicit computation of the state-dependent control updating scheme.**

***Index Terms*—MPC, Certification, Real-time, Stability.**

I. Introduction

Modern control paradigms such as Model Predictive Control [10], Moving-Horizon Observers [1] or adaptive identification of varying models [16] to cite but few issues involve the real-time, on-line solution of constrained optimization problems. In such applications, the output of the optimizer (namely the sub-optimal solution of the optimization problem) is fed to some neighboring modules in order to achieve some engineering tasks. The quality of the global task may strongly depend on the quality of the sub-optimal solution and the frequency with which it can be updated by the optimizer and since this solution has to be delivered in finite and probably short time, it is important to be able to precisely link the quality of the suboptimal solution to the available computation time for a predefined embedded computation power. When the latter is not yet defined, such insight enables to choose the appropriate computational power given the required quality of the sub-optimal solution.

The last few years witnessed an increasing interest in the certification issue [14], [9], [5]. These almost simultaneous works proposed certification bounds for fast gradient-based iterations [13], [12] applied to Quadratic Programming (QP) problems involving only simple constraints that enable easy projection on the admissible set. Otherwise, the iterations that are needed to perform the projection have to be counted as well and certified with some associated lower bounds which would invalidate the relevance of the proposed bounds.

It is not surprising that recent certification-related results concerned fast gradient-based iterations. This is because the simplicity of this iteration and the associated low computational cost have been rapidly identified as appealing properties in the real-time context which is the very reason for which certification results were required.

M. Alamir is with CNRS/Gipsa-lab, Control Systems Dept., University of Grenoble, France e-mail: mazen.alamir@grenoble-inp.fr (http://www.mazenalamir.fr).

Regarding the other alternatives, active set iterations [8], while computationally efficient and while showing a provably finite number of iterations to converge (for QP problems), seem to resist to the derivation of convergence rates which makes impossible the computation of certification bounds. As for interior point methods [7], [15], [4], certification bounds exist [11] but seem to be systematically over pessimistic [14]. Nevertheless, for many problems, it might still be more appropriate to use these efficient although uncertified or pessimistically certified algorithms rather than to use a slow certified iterations. The *right* choice is problem-dependent.

The first part of this paper belongs to the family of works that address the derivation of certification bounds for fast gradient-based iterations in the presence of general constraints. This is motivated by the nice properties mentioned above, namely the reduced complexity of the single associated iteration that enables the use of extremely short updating period. As it has been recently shown [2], [3], [6], this last property may compensate the drawback of potentially higher number of iterations when compared to some alternative methods, especially in uncertain context (which includes perfectly known systems under unpredictable set-point dynamics). In such situations, as underlined by [13], it is important to distinguish between the concepts of *analytical complexity* which involves only the number of iterations (regardless of their inherent computational cost) and the *arithmetical complexity* which accounts for the total number of elementary operations until convergence which is obviously the appropriate indicator in real-time context and this is precisely why fast gradient is an interesting option.

The second part of the paper proposes a general framework to explicitly account for the arithmetical complexity by including the computation time for a single iteration in the overall convergence analysis and trade-off handling. This feature if absent from recent works on the certification issue such as [14] where the number of iteration is induced from the required precision on the solution and the corresponding number of iteration is derived as a consequence. This argumentation suggests that provided that one uses sufficiently high number of iterations, convergence of the real-time MPC will be guaranteed. This paper shows that this argument is generically erroneous and that in realistic situations, the appropriate updating period shows lower and upper bounds

beyond which stability can no more be guaranteed.

More precisely, the contribution of the present paper lies in the following items:

1) it gives a certification bound for the fast gradient algorithm when applied to solve a general (not necessarily simple bounded) convex optimization problems by means of a penalty approach. The number of iterations needed to achieve a prescribed level of precision on the optimal cost and a prescribed level of precision on the satisfaction of the soft constraints (while the hard constraints are fully satisfied) is given as a function of the problem's characteristics.

2) it shows how the certification bound so obtained can be used in the framework of real-time MPC in order to assess the practical asymptotic stability of the closed-loop performance under a state-dependent control updating period. The latter is computed based on some key properties of the MPC formulation. This second part while using the results of the first part has a general scope and can be applied to any available algorithm with computable certification bound.

This paper is organized as follows: Section II defines the class of optimization problems addressed in the paper together with the associated definitions and notation. Section III states the working assumptions and gives some preliminary results that are used in the next sections. The algorithm and the associated certification bounds are presented in section IV with instantiation to the specific case of QP problems. The use of the certification bounds in real-time MPC implementation through state-dependent control updating period is proposed in section V and the concrete computation of the parameters involved in the expressions is shown for the specific case of linear MPC. Finally, the whole scheme is illustrated through the MPC-based tracking problem for a quadruple integrators under state and control constraints. For the sake of clarity, all the technical proofs are gathered in appendix A except those that can be given in few words.

II. Problem Statement

Consider the following optimization problem in the decision variable p:

$$\min_{p\in\mathbb{R}^{n_p}} f_0(p) \quad | \; c_i(p)\leq 0 \quad \forall i\in I_h\cup I_s := \{1,\dots,n_c\} \tag{1}$$

where I_s and I_h are the disjoint subsets of $\{1,\dots,n_c\}$ that define a partition of the set of constraints into soft and hard constraints respectively. $f_0(\cdot)$ is the cost to be minimized while $c_i : \mathbb{R}^{n_p}\rightarrow\mathbb{R}$ defines the i-th inequality constraint. Note that saturation constraints on p are supposed to be included in the set of inequality constraints. It is assumed that f_0 and c_i are differentiable for all i.

The algorithm proposed in this paper invokes the following penalty induced augmented cost:

$$f(p) := f_0(p) + \rho\times\psi(p) \tag{2}$$

where ρ is called the penalty parameter while $\psi:\mathbb{R}^{n_p}\rightarrow\mathbb{R}_+$ is the constraints induced cost given by:

$$\psi(p) := \sum_{i\in I_s}\left[\max\{0,c_i(p)\}\right]^2 + \sum_{i\in I_h}\left[\max\{0,c_i(p)+\varepsilon_\psi\}\right]^2$$

For a given paire $\bar\varepsilon := (\varepsilon_0,\varepsilon_\psi)$ of strictly positive reals, a candidate value p is called an $\bar\varepsilon$-suboptimal solution of (1) if the following two conditions hold:

$$|f_0(p)-f^{opt}|\leq\varepsilon_0 \quad\text{and}\quad \psi(p)\leq\varepsilon_\psi^2 \tag{3}$$

where f^{opt} denotes the optimal value of (1).

The relevance of the second constraint in (3) lies in the fact that when satisfied, this constraint implies that all the hard constraints are rigorously satisfied while the maximum violation of any soft constraint is lower than ε_ψ.

The first aim of the present paper is to derive the necessary relations that enable for a given precision $\bar\varepsilon$ to choose the appropriate penalty coefficient ρ and the stopping condition for the fast gradient iteration to be used in the unconstrained minimization of the cost function f defined by (2). Moreover, the bound on the minimum number of iterations that guarantees an $\bar\varepsilon$-suboptimal solution to the original problem is derived. This is done in sections III and IV.

The second aim is to show that this certification result (or any similar one for possibly another algorithm) can then be used to design a real-time constrained MPC implementation in which a state-dependent control updating period is used to yield certified convergence properties. This is done in section V.

The results are proved in a rather general convex settings and for both goals, the expressions enabling the parameters involved in the statements of the results to be computed are explicitly given in the specific case of QP problems and linear MPC design.

III. Assumptions and preliminary results

A. Definitions and Notation

In what follows, $f^{'}(p)$, $f_0^{'}(p)$ and $\psi^{'}(p)$ denote the gradients of the functions w.r.t the decision variable p. The euclidien norm of $f^{'}(p)$ is denoted by $g(p)=\|f^{'}(p)\|$. For a scalar continuously differentiable function ℓ defined on $\mathbb{R}^n$, the notation $\ell\in\mathcal{S}_\mu^1$ states that ℓ is a μ-strongly convex function, namely for all (p_1,p_2):

$$\ell(p_2)\geq\ell(p_1)+\langle\ell^{'}(p_1),p_2-p_1\rangle+\frac{\mu}{2}\|p_2-p_1\|^2 \tag{4}$$

where μ is called the convexity parameter of ℓ [13]. Similarly, the notation $\ell\in\mathcal{F}_L^1$ indicates that the continuously differentiable function ℓ satisfies for all (p_1,p_2):

$$\ell(p_2)\leq\ell(p_1)+\langle\ell^{'}(p_1),p_2-p_1\rangle+\frac{L}{2}\|p_2-p_1\|^2 \tag{5}$$

When ℓ satisfies both (4)-(5), the notation $\ell\in\mathcal{S}_{\mu,L}^1$ is used. The set $\mathcal{C}$ denotes the set of singular points of $f(\cdot)$, namely

the set of p such that $g(p)=0$. Given a subset $\mathcal{A}\subset\mathbb{R}^{n_p}$, the notation $d(p,\mathcal{A})$ refers to the distance from p to $\mathcal{A}$, namely $d(p,\mathcal{A}):=\min_{z\in\mathcal{A}}\|z-p\|$. The short notation $d(p):=d(p,\mathcal{C})$ is used for the specific set $\mathcal{C}$. The set $\mathcal{A}_{\psi=0}$ is the set of p such that $\psi(p)=0$. Given a bounded subset $\mathbb{P}$, $\delta_{\mathbb{P}}$ denote the radius of $\mathbb{P}$ namely $\delta_{\mathbb{P}}:=\sup_{(x_1,x_2)\in\mathbb{P}^2}\|x_1-x_2\|$. For a compact set $\mathbb{X}$, the notation $\varrho(\mathbb{X})$ denotes the maximum norm of elements in $\mathbb{X}$, namely $\varrho(\mathbb{X}):=\sup_{x\in\mathbb{X}}\|x\|$.

B. Working Assumptions

Assumption 3.1: The cost function value $f_0(p)$ is nonnegative for all p.

This assumption can be made satisfied by adding sufficiently high positive constant. It is quite common in MPC context where the cost function refers quite often to the integral of the tracking error that is added to some positive terminal term.

Assumption 3.2: There are two reals $L_0\geq 0$ and $L_\psi\geq 0$ such that $f_0\in\mathcal{F}^1_{L_0}$ and $\psi\in\mathcal{F}^1_{L_\psi}$

Assumption 3.3: There is $\mu_0>0$ such that $f_0\in\mathcal{S}^1_{\mu_0}$. Moreover, Ψ is convex.

Note that this assumption implies that $f\in\mathcal{S}^1_{\mu_0}$ and that there is a unique critical point for f which is denoted hereafter by $p^*\in\mathcal{C}$. therefore according to the definition of $d(p)$, one has $d(p):=\|p-p^*\|$.

In what follows, the notation p_u and p_a refer to two vectors such that:

$$p_u:=\min_{p\in\mathbb{R}^p} f_0(p)\quad ;\quad \psi(p_a)\leq 0 \tag{6}$$

namely, p_u is the unconstrained minimum of f_0 while p_a is any admissible point. Having p_a, the following definition can be stated since f_0 is supposed to be continuously differentiable and because $f_0\in\mathcal{S}^1_{\mu_0}$ [Assumption 3.3]:

Definition 3.1: Define D_0 by:

$$D_0:=\sup_{f_0(p)\leq f_0(p_a)}\|f_0^{'}(p)\|\geq 0 \tag{7}$$

Remark 3.1: In fact, the knowledge of the admissible point p_a is only required to compute D_0. therefore, if an upper bound of D_0 can be found, the knowledge of p_a is not mandatory. This is clearly shown in section IV-C in the specific case of QP problems [see inequality (23)] . This is crucial since in the MPC context the constraints are state dependent and it may become cumbersome to compute p_a for each current state.

The next assumption concerns the behavior of the penalty map outside the admissible set.

Assumption 3.4: There is $\beta>0$ such that the following inequality:

$$\psi(p)\geq\beta\times\Bigl[d(p,\mathcal{A}_{\psi=0})\Bigr]^2 \tag{8}$$

holds for all p. ◇

The expressions of the parameters L_0, L_ψ, μ_0, D_0 and β in the specific case of quadratic cost f_0 and affine in p constraints c_i are given in section IV-C.

C. Preliminary results

In this section some preliminary results are stated. For better readability, all the proofs are given in the appendix. The first result gives a property of the gradient of f_0 at the stationary point p^*:

Lemma 3.1: The following inequality holds

$$\|f_0^{'}(p^*)\|\leq D_0 \tag{9}$$

PROOF. See Appendix A.

The following result characterizes the behavior of the penalty term ψ in terms of the penalty coefficient ρ:

Lemma 3.2: **If** $\rho>L_0/\beta$ **then** the following inequality:

$$\psi(p)\leq\frac{L_\psi}{2}\Bigl[d(p)+\frac{\kappa_0}{\sqrt{\rho}}\Bigr]^2\ \text{where}\ \kappa_0:=\frac{2L_0}{\beta}\sqrt{\frac{2}{\mu_0}\psi(p_u)} \tag{10}$$

holds for all p. In particular, for p^* one has:

$$\psi(p^*)\leq\frac{L_\psi\kappa_0^2}{2\rho} \tag{11}$$

PROOF See Appendix B.

Note that Lemma 3.2 quantifies how increasing ρ leads to a smaller constraint violation depending on the amount of violation $\psi(p_u)$ at the unconstrained minimum p_u of f_0.

The following corollary gives a bound on the difference in the cost f_0 evaluated at the unconstrained optimum p^* of $f=f_0+\rho\psi$ and the true optimal cost as a function of constraint violation:

Lemma 3.3: Let p^{opt} be the optimal solution of the original problem (1). p^* the unconstrained minimum of f. The following inequality holds:

$$|f_0(p^{opt})-f_0(p^*)|\leq D_0\left[\frac{\psi(p^*)}{\beta}\right]^{\frac{1}{2}}+\frac{L_0}{2}\left[\frac{\psi(p^*)}{\beta}\right] \tag{12}$$

PROOF. See Appendix C.

Using Lemma 3.3 one can prove the following result:

Corollary 1: **If** the following inequality holds:

$$\left[\frac{\psi(p^*)}{\beta}\right]^{\frac{1}{2}}\leq Z_1(\epsilon):=\frac{D_0}{L_0}\left[\left(1+\frac{2L_0}{D_0^2}\epsilon\right)^{\frac{1}{2}}-1\right] \tag{13}$$

then the stationary solution p^* satisfies:

$$|f_0(p^{opt})-f_0(p^*)|\leq\epsilon \tag{14}$$

PROOF. This can be easily obtained after noticing that the r.h.s of (12) is a second order polynomial in $\sqrt{\psi(p^*)/\beta}$. Writing that this polynomial is equal to ϵ and solving for it gives the result. □

Note however that p^* is never reached exactly. Instead, the fast gradient iteration will be used to reach an iterate p that is close to p^*. Now since the available certification bounds on the fast gradient iterations concern the guaranteed value of $|f(p^*) - f(p)|$ while the $\bar{\varepsilon}$-suboptimality is defined in terms of the original cost f_0, the following lemma gives a link between these two indicators:

Lemma 3.4: The following implication holds for all ϵ:

$$\Big\{|f(p) - f(p^*)| \le \epsilon\Big\} \Rightarrow$$
$$\Big\{|f_0(p) - f_0(p^*)| \le D_0 \left[\frac{2\epsilon}{\mu_0}\right]^{\frac{1}{2}} + \frac{L_0}{2}\left[\frac{2\epsilon}{\mu_0}\right]\Big\} \tag{15}$$

PROOF. See Appendix D.

Here again, Lemma 3.4 gives the condition on the precision ϵ_1 required on f in order to induce a precision ϵ_2 on f_0, namely:

Corollary 2: **If** p is such that $|f(p) - f(p^*)| \le \epsilon_1$ with

$$\left[\frac{2\epsilon_1}{\mu_0}\right]^{\frac{1}{2}} \le Z_1(\epsilon_2) \tag{16}$$

where Z_1 is the function defined by (13) **then**, one has $|f_0(p) - f_0(p^*)| \le \epsilon_2$.

PROOF. Use the same arguments as before since the r.h.s of (15) involves the same polynomial as in (12). □

The certification bound of the fast gradient needs an upper bound on the distance between the initial guess p and the minimizer of f, namely p^*. The following lemma gives such an upper bound in terms of the value of the function f at the initial guess p:

Lemma 3.5: The following inequality is satisfied for all p:

$$\|p^* - p\| \le \left[\frac{2f(p)}{\mu_0}\right]^{\frac{1}{2}} =: r(p) \tag{17}$$

PROOF. This is a direct consequence of the inclusion $f \in \mathcal{S}^1_{\mu_0}$ and the fact that f_0 (and hence f) is positive. □

IV. THE ALGORITHM

A. Recalls on the Fast Gradient iteration

The fast gradient algorithm proposed in [13] is commonly used to perform unconstrained minimization of a function $f \in \mathcal{S}_{\mu,L}$. It is briefly recalled through Algorithm 1 for which the following convergence result holds

Algorithm 1 $[p_N, q_N, \alpha_N] = F^{(N)}(p_0, q_0, \alpha_0)$

for $i = 1 : N$ **do**
  $p_{i+1} \leftarrow q_i - f^{'}(q_i)/L$
  Compute $\alpha_{i+1} \in (0,1)$ solution of $\alpha_{i+1}^2 = (1 - \alpha_{i+1})\alpha_i^2 + \mu_0\alpha_{i+1}/L$
  $\beta_i \leftarrow (\alpha_i(1-\alpha_i))/(\alpha_i^2 + \alpha_{i+1})$
  $q_{i+1} \leftarrow p_{i+1} + \beta_i(p_{i+1} - p_i)$
end for

Proposition 4.1: ([13], page 80) The successive iterates of Algorithm 1 starting from the initial guess p_0, $\alpha_0 = \sqrt{\mu_0/L}$ and $q_0 = p_0$ satisfy the following inequality:

$$f(p_i) - f(p^*) \le$$
$$\frac{L+\mu_0}{2} \times \min\left\{(1-c)^i, \frac{1}{(1+ic)^2}\right\} \times \|p_0 - p^*\|^2 \tag{18}$$

where $c := \sqrt{\mu_0/L}$ and where p^* stands for the unconstrained minium of f. ◊

The following is a direct consequence of Proposition 4.1:

Corollary 3: If the initial guess satisfies $\|p_0 - p^*\| \le \delta$ then for any $\epsilon > 0$, the integer:

$$\bar{N}(c,\gamma) := \max\left\{0, \min\{\frac{\log(\gamma)}{\log(1-c)}, \frac{1}{c}(\sqrt{\frac{1}{\gamma}} - 1)\}\right\} \tag{19}$$
$$\text{where} \quad \gamma := 2\epsilon/((L+\mu_0)\delta^2)\ ;\ c = \sqrt{\mu_0/L}$$

is an upper bound of the number of iterations N needed by Algorithm 1 to deliver a sub-optimal solution p_N satisfying $|f(p_N) - f(p^*)| \le \epsilon$. ◊

PROOF. Inject $\|p_0 - p^*\| \le \delta$ in (18) and impose that the r.h.s is $\le \epsilon$. □
Now using the bound on $\|p_0 - p^*\| \le r(p_0)$ given by (17), the following result follows:

Corollary 4: Given any initial value p_0, let $\gamma_0 := \epsilon\mu_0/[(L+\mu_0)f(p_0)]$ then $N(c,\gamma_0)$ is an upper bound of the number of iterations N needed by Algorithm 1 to deliver a sub-optimal solution p_N satisfying $|f(p_N) - f(p^*)| \le \epsilon$. ◊

B. The Proposed Algorithm

The proposed algorithm involves the quantities defined by (20)-(21) that depend on:

- the problem's intrinsic properties $(\mu_0, L_0, L_\psi, \beta, D_0)$
- the unconstrained solution-dependent parameter κ_0 [see (10)]
- the desired precision pair $\bar{\varepsilon} := (\varepsilon_0, \varepsilon_\psi)$

$$\rho_1 := \frac{2L_\psi\kappa_0^2}{\varepsilon_\psi^2}$$
$$\rho_2 := \frac{L_\psi\kappa_0^2}{2\beta Z_1^2(\varepsilon_0/2)} \tag{20}$$
$$\rho_3 := L_0/\beta$$

$$\eta_1 := \frac{\mu_0}{2}Z_1^2(\frac{\varepsilon_0}{2})$$
$$\eta_2 := \frac{\mu_0\varepsilon_\psi^2}{4L_\psi} \tag{21}$$

These quantities are used in Algorithm 2 below:

Algorithm 2 $\hat{p}^* = A(p_0, \bar{\varepsilon} := (\varepsilon_0, \varepsilon_\psi))$

1: $\alpha_0 = (\mu_0/L)^{\frac{1}{2}}$, $q_0 := p_0$
2: $\rho = \max\{\rho_1, \rho_2, \rho_3\}$
3: $\eta = \min\{\eta_1, \eta_2\}$
4: $c = \sqrt{\mu_0/L}$
5: $\gamma_0 = \eta\mu_0/[(L+\mu_0)f_0(p_0)]$
6: $N_{max} = \bar{N}(c, \gamma_0)$
7: $g_{min} = \mu_0\sqrt{2\eta/L}$
8: again=true
9: **while** (again) **do**
10: $\quad [p_{i+1}, q_{i+1}, \alpha_{i+1}] = FG^{(1)}(p_i, q_i, \alpha_i)$
11: $\quad$ **if** $[(i \geq N_{max})$ or $(g(p_i) \leq g_{min})]$ **then**
12: $\qquad$ again=0
13: $\quad$ **else**
14: $\qquad i = i + 1$
15: $\quad$ **end if**
16: **end while**
17: $\hat{p}^* = p_i$

The following result gives a certification bound on the number of iterations needed by Algorithm 2 to achieve an $\bar{\varepsilon}$-suboptimal solution of the original problem.

Proposition 4.2: Let be given a precision pair $\bar{\varepsilon} := (\varepsilon_0, \varepsilon_\psi)$, an initial guess p_0. Let $\gamma_0 := \eta\mu_0/[(L+\mu_0)f_0(p_0)]$ where $\eta := \min\{\eta_1, \eta_2\}$ with the η_is given by (21). The algorithm in which $\rho = \max\{\rho_1, \rho_2, \rho_3\}$ is used with the ρ_is defined by (20) involves at most $\bar{N}(c, \gamma_0)$ unconstrained fast gradient elementary iterations before it delivers an estimate $\hat{p}^*$ that is an $\bar{\varepsilon}$-suboptimal solution of the original constrained optimization problem (1).

PROOF. See Appendix E.

In the remainder of the paper, the maximum number of iterations that guarantee the precision as expressed in Proposition 4.2 is denoted by:

$$N(p_0, \varepsilon_0, \varepsilon_\psi) := \bar{N}(c, \gamma_0) \tag{22}$$

as the arguments of N completely determine c and γ_0.

C. Case of Quadratic Programming (QP) problems

Here, the expressions of L_0, L_ψ, μ_0, D_0 and β are given in the specific case of QP problems where the cost function and the constraints take the form:

$$f_0(p) = \frac{1}{2}p^T Hp + F^T p + s_0 \quad ; \quad c_i(p) = A_i p - B_i$$

In this case, Assuming that s_0 is such that assumption 3.1 holds, it is straightforward that Assumptions 3.2 and 3.3 holds with $L_0 = \lambda_{max}(H)$, $L_\psi = \sigma_{max}(A)$ and $\mu_0 = \lambda_{min}(H)$. Moreover, one has $p_u := -H^{-1}F$. Now according to remark 3.1, p_a is not needed provided that an upper bound for D_0 can be derived. This is the aim of the following proposition:

Proposition 4.3: Provided that the set of inequalities $Ap \leq B$ implies the condition $p \in \mathbb{P}$, the following inequality holds:

$$D_0 \leq \quad [\lambda_{max}(H)] \cdot \bar{p} + \|F\| \tag{23}$$

where

$$\bar{p} \quad := \quad \frac{\|F\| + \sqrt{\|F\|^2 + 2\lambda_{min}(H)\left[\bar{f}\right]}}{\lambda_{min}(H)} \tag{24}$$

in which

$$\bar{f} := \frac{1}{2}\lambda_{max}(H)\left[\varrho(\mathbb{P})\right]^2 + \|F\| \cdot \varrho(\mathbb{P}) \tag{25}$$

PROOF. See Appendix F.

Assumption 3.4 is satisfied with $\beta := \sigma_{min}(A)$ which is the lowest non zero singular value of the constraints matrix A. The coefficient κ_0 involved in lemma 3.2 and the expressions (20)-(21) used to compute ρ and η is obtained using the values of L_0, β, μ_0 and p_u mentioned above.

NUMERICAL EXPERIMENTS In order to check the validity of the certification bound $N(p_0, \varepsilon_0, \varepsilon_\psi)$, 500 random QP problems have been generated with $n = 10$ decision variables and $n_c = 20$ constraints. More precisely, $H := CC^T + \sigma\mathbb{I}$ is used where $C \in \mathbb{R}^{n\times 1}$ and $\sigma \in [10^{-3}, 1]$, F and s_0 has been computed so that the cost is $\|p - p_u\|_H^2 + 1$ where p_u is randomly generated. The constraints matrices $A \in \mathbb{R}^{n\times n_c}$ and $B \in \mathbb{R}^{n_c}$ has been randomly generated so that a feasible solution exists. The precision $\epsilon_\psi = 10^{-2}$ has been used while ε_0 has been systematically taken equal to 1% of the true optimal cost that is obtained by QUADPROG-MATLAB solver. The initial guess is systematically taken equal to 0 as one might use in cold start MPC context.

The results are shown in Figure 1 where the histogram over the 500 trials of the ratio between the effectively needed number of iterations N and the maximal computed certification bound N_{max} (step 6 of Algorithm 2) is plotted. The results suggest that for this class of QP problems, the bounds is not that conservative and that since some scenarios lead to a ratio between 0.5 and 0.6, as far as certification is needed, it cannot be strongly reduced.

V. APPLICATION TO REAL-TIME MPC

In this section, it is assumed that a certification bound $N(p_0, \varepsilon_0, \varepsilon_\psi)$ is given for some algorithm. Based on such a bound, a real-time MPC implementation framework is proposed using a state-dependent control updating period leading to provable practical convergence. It is therefore important to underline that the results of this section does not necessarily relate to the use of the fast-gradient algorithm as they can apply to any algorithm for which a certification can be associated that depends on the initial guess p_0 and some required precision pair (ε_0 and ε_ψ) in the sense of (3).

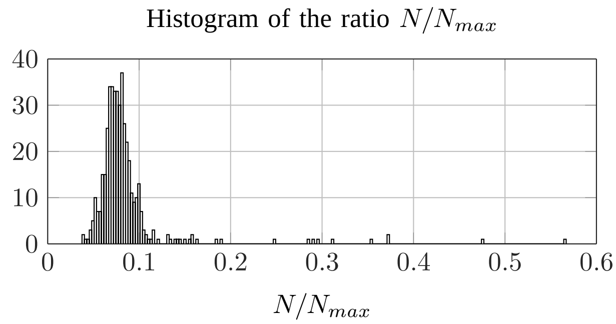

Fig. 1. Histogram showing the statistics of the ratio N/N_{max} between the effectively needed number of iterations N and the certification bound N_{max} computed from the theory when using the numerical experiments described in section IV-C.

A. Definition, notation and working assumptions

In this section, a set of assumptions are stated. Not all of them are used in all the subsequent results. That is why in the statement of each result, the assumptions that are needed are explicitly mentioned.

In MPC framework, the controller disposes of a model of the form

$$\dot{x} = F(x, u) \quad (x, u) \in \mathbb{R}^n \times \mathbb{R}^{n_u} \tag{26}$$

where the following assumption is used regarding the definition of the vector x:

Assumption 5.1: The state vector x involved in (26) gathers the physical state of the system together with the current set-point and current estimation of the disturbance. The model also incorporates the assumption on the future behavior of these exogenous variables.

We consider that the future control profiles are parametrized through a finite dimensional vector p of degrees of freedom such that at each instant t, the future profile depends on $p(t)$ according to:

$$u(t+s) := \mathcal{U}(s, p(t)) \quad s \in [0, T] \tag{27}$$

where $\mathcal{U}$ is some predefined map and T is the prediction horizon.

Since the MPC has to be computed based on the prediction of the future state (in the sense of Assumption 5.1), the following assumption is needed to characterize the state prediction error:

Assumption 5.2: For each compact set $\mathbb{C}$ to which belongs the pair $(p(t), x(t))$, the prediction $\hat{x}(t+\tau)$ of the future state starting from $x(t)$ and under the control profile $\mathcal{U}(\cdot, p(t))$ can be affected by an error satisfying

$$\|\hat{x}(t+\tau) - x(t+\tau)\| \leq E^0_{\mathbb{C}} + E^1_{\mathbb{C}} \times \tau \tag{28}$$

Note that $E^0_{\mathbb{C}}$ in (28) accommodates for unpredictable set-point changes while $E^1_{\mathbb{C}}$ accommodates for the presence of disturbances that affects the input of some integrator in the system or for the presence of unpredictable time-varying set-point.

The cost function is defined at instant t based on the knowledge of the state $x(t)$ (including the current value of the set point and the disturbance estimation and prediction). This leads to a constrained optimization problem of the form (1) in which both f_0 and c_i are dependent on the current value $x(t)$ of the state, namely:

$$f_0(p, x(t)) \quad ; \quad \psi(p, x(t))$$

Consequently, the call of Algorithm 2 as well as the bound (22) on the number of iterations must now incorporate the state $x(t)$ as an argument, namely:

$$\hat{p}^* = A(p_0, \varepsilon_0, \varepsilon_\psi, x) \quad ; \quad N(p_0, \varepsilon_0, \varepsilon_\psi, x) \tag{29}$$

In order to use the results of the preceding section, one needs to assume that for all x, there are positive reals $L_0(x)$, $L_\psi(x)$ and $\beta(x)$ and a strictly positive $\mu_0(x) > 0$ that play the roles of L_0, L_ψ, β and μ_0 as defined in the preceding section.

Now if for some reasons, one knows that the pair (p_0, x) involved in (29) belongs to some compact set $\mathbb{C} := \mathbb{P} \times \mathbb{X}$, then one can obtain a certification bound that depends only on the precision parameters $\bar{\varepsilon} := (\varepsilon_0, \varepsilon_\psi)$, namely:

$$N_{\mathbb{C}}(\varepsilon_0, \varepsilon_\psi) := \max_{(p,x)\in\mathbb{C}} N(p, \varepsilon_0, \varepsilon_\psi, x) \tag{30}$$

Moreover, the following result shows that the bound $N_{\mathbb{C}}(\varepsilon_0, \varepsilon_\psi)$ can be computed through static optimization steps involving the functions f_0 and ψ:

Proposition 5.1: Let a compact set $\mathbb{C} := \mathbb{P} \times \mathbb{X}$ be given. the bound $N_{\mathbb{C}}(\epsilon_0, \epsilon_\psi)$ defined by (30) can be computed by the following steps:

1) Compute ψ^{max} according to:
$$\psi^{max} := \max_{x\in\mathbb{X}} \Big\{ \psi(p_u, x) \mid f_0^{'}(p_u, x) = 0 \Big\} \tag{31}$$
2) Compute L_0, L_ψ as the maximum of $L_0(x)$ and $L_\psi(x)$ over $x \in \mathbb{X}$
3) Compute β and μ_0 as the minimums of $\beta(x)$ and $\mu_0(x)$ over $x \in \mathbb{X}$
4) Compute $\kappa_0^{max} := \dfrac{2L_0}{\beta}\sqrt{2\psi^{max}/\mu_0}$
5) Compute ρ^{max} using (20) in which κ_0^{max} replaces κ_0
6) Compute $\eta^{min} := \min\{\eta_1, \eta_2\}$ where the η_i are computed by (21) in which ρ^{max} replaces ρ.
7) Compute $f_0^{max} := \max_{(p,x)\in\mathbb{C}} f_0(p, x)$
8) Compute $\gamma_0^{min} := \eta^{min}\mu_0/[(L(\rho^{max}) + \mu_0)f_0^{max}]$
9) Compute $c^{min} := \sqrt{\mu_0/L(\rho^{max})}$

Finally compute the desired quantity:

$$N_{\mathbb{C}}(\varepsilon_0, \varepsilon_\psi) := \bar{N}(c^{min}, \gamma_0^{min}) \tag{32}$$

where $\bar{N}$ is defined by (19).

PROOF. Straightforward as the computation systematically takes the worst case towards the increase of N. $\square$

In section V-C, Explicit computation of all the quantities involved in Proposition 5.1 is given for the specific case of state-dependent QP optimization problems that arise in the linear MPC context.

It is also assumed that the cost function f_0 is proper in both p and x in the following sense:

Assumption 5.3: For any positive real $\phi > 0$, there is a compact set $\mathbb{C}_\phi$ such that the following implication holds:

$$\Big\{f_0(p,x) \le \phi\Big\} \Rightarrow \Big\{(p,x) \in \mathbb{C}_\phi\Big\} \tag{33}$$

Regarding the dependence of f_0 and ψ on x, the following assumption is considered:

Assumption 5.4: For any compact set $\mathbb{C}$, there are positive real $K_{\mathbb{C}}^0, K_{\mathbb{C}}^\psi > 0$ such that :

$$\|f_0(p,x_1) - f_0(p,x_2)\| \le K_{\mathbb{C}}^0 \cdot \|x_1 - x_2\| \tag{34}$$
$$\|\psi(p,x_1) - \psi(p,x_2)\| \le K_{\mathbb{C}}^\psi \cdot \|x_1 - x_2\| \tag{35}$$

for all $(p,x_1),(p,x_2) \in \mathbb{C}$.

A typical formulation of $f_0(p,x_0)$ in MPC is given by:

$$\begin{aligned} f_0(p,x_0) &:= \Omega(\bar{x}(T,p,x_0)) + \int_0^T \ell(\bar{x}(s,p,x_0),p,s)ds \\ &=: \Omega(\bar{x}(T,p,x_0)) + \int_0^T \bar{\ell}(s,p,x_0)ds \end{aligned} \tag{36}$$

where $\bar{x}(s,p,x_0)$ is the predicted state value at instant s starting from x_0 at instant 0.

Regarding the formulation of the MPC, the following (commonly satisfied) assumption is needed in the sequel:

Assumption 5.5: The MPC formulation is based on a cost function of the form (36) with the necessary constraints that make the following inequality satisfied:

$$\begin{aligned} &f_0(p^{opt}(t+\tau),\hat{x}(t+\tau)) - f_0(p^{opt}(t),x(t)) \le \\ &\le -\Delta(\tau,x(t)) := -\int_0^\tau \bar{\ell}(s,p^{opt}(t),x(t))ds \end{aligned} \tag{37}$$

where $p^{opt}(t)$ is the optimal solution of the problem defined for the state $x(t)$ while $p^{opt}(t+\tau)$ is the optimal solution of the problem defined by the predicted future state $\hat{x}(t+\tau)$ starting from $x(t)$ under the optimal control $\mathcal{U}(\cdot,p^{opt}(t))$ that is applied on the interval $[t,t+\tau]$.

Note that $p^{opt}(t)$ does not appear as an argument of Δ since $p^{opt}(t)$ is assumed to be uniquely determined by $x(t)$.

Remark 5.1: Note that the inequality (37) is satisfied only for the ideal predicted future state $\hat{x}(t+\tau)$ since otherwise the bad knowledge of uncertainties and/or the set-point changes may invalidate the inequality if the true value $x(t+\tau)$ of the state is used.

Remark 5.2: Note that inequality (37) is commonly satisfied in the standard provably stable MPC formulations. Moreover, the r.h.s $\Delta(\tau,x(t))$ is generally exhibited through the corresponding stability proof (see [10]).

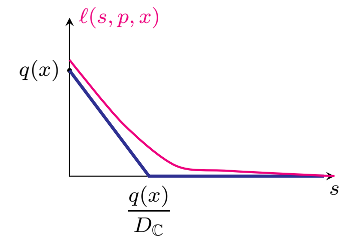

Fig. 2. Illustration of Assumption 5.6.

Regarding the penalty function ℓ, the following assumption is used:

Assumption 5.6: [Figure 2] For any compact set $\mathbb{C}$, there is a positive real $D_{\mathbb{C}} > 0$ and a positive function $q(\cdot)$ such that :

$$\bar{\ell}(s,p,x) \ge \max\{0, q(x) - D_{\mathbb{C}} s\} \tag{38}$$

for all $(p,x) \in \mathbb{C}$.

Note that condition (38) simply states that with bounded control, there is a limitation on the speed with which the state can be steered to the desired region. With this respect, $q(x)$ is simply a state dependent term in ℓ that expresses how far does x lie from the desired region. This notation enables many situations to be handled as x includes set-point definition and therefore, mesures of the difference between the physical state of the system and their desired value can take the simple form expressed by $q(x)$.

Finally, the following assumption is used to characterize the available computational facility:

Assumption 5.7: The system is controlled with a computational facility that performs a single elementary iteration of the fast gradient (step 9 of Algorithm 2) in τ_c time units.

Note that if another certified algorithm than the fast gradient is used, τ_c used hereafter denotes the time necessary to perform a single iteration of that specific algorithm.

B. Certified MPC by state-dependent updating period

Assume that a scheme is based on the iterative on-line definition of a sequence of updating instants and a sequence of precision parameters denoted by:

$$t_{k+1} = t_k + \tau_k \quad ; \quad \big\{\varepsilon_0^{(k)}, \varepsilon_\psi^{(k)}\big\}_{k=0}^\infty \tag{39}$$

which are linked through the definition of the updating periods τ_k according to:

$$\tau_k := \tau_c \times N_{\mathbb{C}}\big(\varepsilon_0^{(k+1)}, \varepsilon_\psi^{(k+1)}\big) \tag{40}$$

where $\mathbb{C}$ is some compact subset of $\mathbb{R}^{n_p} \times \mathbb{R}^n$ and τ_c is the computation time needed for a single fast gradient iteration (see Assumption 5.7).

More precisely, given the current state $x(t_k)$ and a control $\mathcal{U}(\cdot,\hat{p}^*(t_k))$ that is applied during the sampling period $[t_k,t_{k+1}]$, Algorithm 2 is used to compute the control parameter $\hat{p}^*(t_{k+1})$ (that is to be applied during the next sampling period) with the hot start $[\hat{p}^*(t_k)]^{+\tau_k}$ and the precision parameters $(\varepsilon_0^{(k+1)},\varepsilon_\psi^{(k+1)})$. Note that by the very definition (40) of τ_k, the value of the control parameter $\hat{p}^*(t_{k+1})$ that is obtained by Algorithm 2 before t_{k+1} necessarily meets the precision requirements, namely:

$$\begin{aligned}&f_0(\hat{p}^*(t_{k+1}),\hat{x}(t_{k+1}))-f_0(p^{opt}(t_{k+1}),\hat{x}(t_{k+1}))\le\varepsilon_0^{(k+1)}\\&c_i(\hat{p}^*(t_{k+1}),\hat{x}(t_{k+1}))\le 0 \qquad i\in I_h\\&c_i(\hat{p}^*(t_{k+1}),\hat{x}(t_{k+1}))\le\varepsilon_\psi^{(k+1)} \qquad i\in I_s\end{aligned}\tag{41}$$

Using the first inequality, one can prove the following result:

Lemma 5.1: **If** the following conditions hold

1) τ_k is defined by (40) for some compact set $\mathbb{C}:=\mathbb{P}\times\mathbb{X}$
2) For all k, $[\hat{p}^*(t_k)]^{+\tau_k}\in\mathbb{P}$
3) For all k, $x(t_k)\in\mathbb{X}$
4) Assumptions 5.2, 5.4 and 5.5 are satisfied

then the following inequality holds for all k:

$$\begin{aligned}&f_0(\hat{p}^*(t_{k+1}),x(t_{k+1}))-f_0(\hat{p}^*(t_k),x(t_k))\le\\&\varepsilon_0^{(k)}+K^0_{\mathbb{C}}(E^0_{\mathbb{C}}+E^1_{\mathbb{C}}\tau_k)+\varepsilon_0^{(k+1)}-\Delta(\tau_k,x(t_k))\end{aligned}\tag{42}$$

PROOF. See Appendix G.

Note that the term $f_0(\hat{p}^*(t_k),x(t_k))$ represents the value of the cost function at the effectively *visited* pairs $(\hat{p}(t_k),x(t_k))$. Therefore, the difference expressed in the l.h.s of (42) is relevant for the stability assessment of the resulted truncated MPC implementation. On the other hand, using the definition (40) of τ_k, the r.h.s of (42) can be viewed as a function of the precision pair $(\varepsilon_0^{(k+1)},\varepsilon_\psi^{(k+1)})$. The stability issue is therefore dependent on the possibility to define these precision parameters in such a way that the r.h.s of (42) is negative. This is the aim of the following development.

Since the only negative term in the r.h.s of (42) is $-\Delta(\tau_k,x(t_k))$, we need a lower bound on $\Delta(\tau_k,x(t_k))$. The following straightforward lemma gives such a lower bound:

Lemma 5.2: **If** the following conditions hold:

1) $(\hat{p}^*(t_k),x(t_k))\in\mathbb{C}$
2) Assumption 5.6 is satisfied

then a computable lower bound of the quantity $\Delta(\tau,x(t_k))$ can be obtained by:

$$\Delta(\tau,x(t_k))\ge\Gamma_{\mathbb{C}}(\tau,q(x(t_k)))\tag{43}$$

where $\Gamma_{\mathbb{C}}(\tau,q)$ is given by (see Figure 3):

$$\Gamma_{\mathbb{C}}(\tau,q):=\begin{cases}q\tau-\dfrac{1}{2}D_{\mathbb{C}}\tau^2 & \text{if } \tau\le q/D_{\mathbb{C}}\\ \dfrac{q^2}{2D_{\mathbb{C}}} & \text{otherwise}\end{cases}\tag{44}$$

PROOF. See Appendix H.

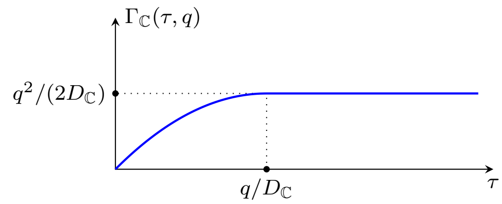

Fig. 3. Evolution of $\Gamma_{\mathbb{C}}(\tau,q)$ involved in Lemma 5.2.

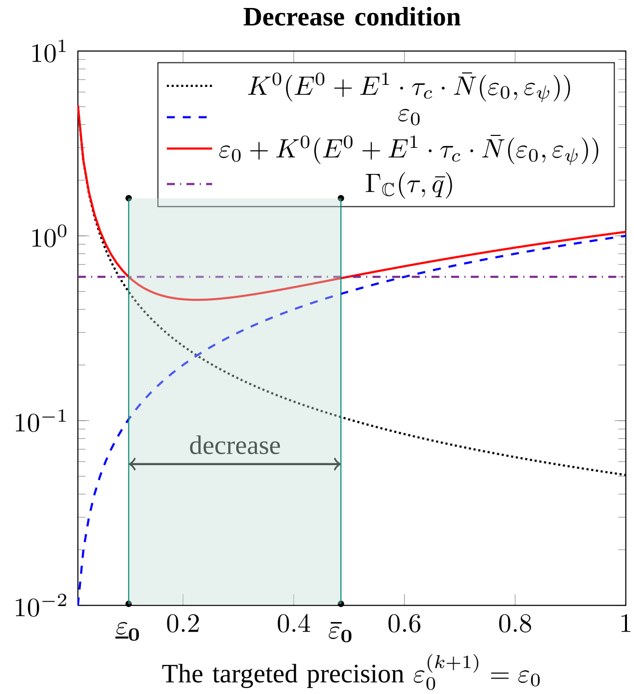

Fig. 4. Typical evolution of the quantities involved in the r.h.s of equation (46) invoked in corollary 5. The decrease of the cost function is possible if there is a tagreted future precision ε_0 for which the red-solid curve lies below the dash-dotted curve.

Using the definition (40) of τ_k and the r.h.s of (43) in (42) the following computable function can be defined:

$$\begin{aligned}&R_{\tau_c}(\varepsilon_0,\varepsilon_\psi,\bar{q}):=K^0_{\mathbb{C}}(E^0_{\mathbb{C}}+\tau_cE^1_{\mathbb{C}}\bar{N}(\varepsilon_0,\varepsilon_\psi))+\varepsilon_0-\\&\Gamma_{\mathbb{C}}(\tau_c\cdot\bar{N}(\varepsilon_0,\varepsilon_\psi),\bar{q})\end{aligned}\tag{45}$$

so that the following corollary of Lemma 5.1 can be stated:

Corollary 5: **If** the following conditions hold

1) The requirements of Lemma 5.1 are satisfied
2) Assumption 5.6 holds
3) $q(x(t_k))\ge\bar{q}$

then the following inequality holds

$$\begin{aligned}&f_0(\hat{p}^*(t_{k+1}),x(t_{k+1}))-f_0(\hat{p}^*(t_k),x(t_k))\le\\&\varepsilon_0^{(k)}+R_{\tau_c}(\varepsilon_0^{(k+1)},\varepsilon_\psi^{(k+1)},\bar{q})\end{aligned}\tag{46}$$

where $R_{\tau_c}(\cdot)$ is defined by (45).

Figure 4 presents a typical situation showing that for a given past achieved precision $\varepsilon_0^{(k)}$, a given computational power

leading to the computation time τ_c and a given precision ε_ψ on the soft constraints satisfaction, either there is no $\varepsilon_0^{(k+1)}$ making the r.h.s of equation (45) invoked in corollary 5 negative or there is an interval of successful values of $\varepsilon_0^{(k+1)}$ which does not contain 0 and which depends on the current value of $q(x(t_k)) = \bar{q}$.

Note that corollary 5 involves quantities that depend on some compact set to which belong all the pair $([\hat{p}^*(t_k)]^{+\tau_k}, \hat{x}(t_{k+1}))$. Using assumption 5.3, it is possible to prove that such compact set is linked to a set of initial conditions for which a certified convergence result can be derived for the resulting real-time MPC. This is stated in the following proposition which is the main contribution of the paper:

Proposition 5.2: Consider a positive real $\phi_0 > 0$ and the corresponding compact subset $\mathbb{C}_{\phi_0} \subset \mathbb{R}^{n_p} \times \mathbb{R}^n$ defined according to assumption 5.3. Let be given a precision $\varepsilon_\psi > 0$ on the soft constraints satisfaction.

If the following conditions hold with $\mathbb{C} = \mathbb{C}_{\phi_0}$:

1) Assumptions 5.2, 5.4, 5.5 and 5.6 are satisfied
2) $\exists \bar{q}_{min} > 0$ and $\gamma_c > 0$ such that the inequality:
$$R_{\tau_c}(\varepsilon_0, \varepsilon_\psi, \bar{q}) \leq -\left[\frac{\gamma_c \bar{q}_{min}^2}{3D_{\mathbb{C}_{\phi_0}}}\right] \tag{47}$$
admits a solution $\varepsilon_0^{sol}(\bar{q}) \in [0, \gamma_c \bar{q}_{min}^2/(2D_{\mathbb{C}_{\phi_0}})]$ for all $\bar{q} \geq \bar{q}_{min}$

then the truncated MPC design based on the adaptive sampling period defined by:
$$\tau_k := \tau_c \times \bar{N}(\varepsilon_0^{sol}(q(x(t_k)), \varepsilon_\psi) \tag{48}$$
steers the system to the set:
$$\mathbb{X}_{min} := \Big\{ x \in \mathbb{R}^n \quad | \quad q(x) \leq \bar{q}_{min} \Big\} \tag{49}$$
provided that the initial condition satisfies:
$$f_0(\hat{p}^*(t_0), x(t_0)) \leq \phi_0 \quad ; \quad \varepsilon_0^{(0)} \leq \frac{\gamma_c \bar{q}_{min}^2}{6D_{\mathbb{C}_{\phi_0}}} \tag{50}$$
Moreover, if the hard constraints depend only on p, then along the closed-loop trajectory, one has:
$$\begin{aligned} \max_{i \in I_h, k \geq 0} [c_i(\hat{p}^*(t_k), x(t_k))] &\leq 0 \\ \max_{i \in I_s, k \geq 0} [c_i(\hat{p}^*(t_k), x(t_k))] &\leq \varepsilon_\psi + \\ &+ K_{\mathbb{C}_{\phi_0}}^\psi \cdot (E_{\mathbb{C}_{\phi_0}}^0 + + E_{\mathbb{C}_{\phi_0}}^1 \tau_k) \end{aligned} \tag{51}$$
PROOF. See Appendix I.

C. Case of linear MPC

Linear MPC formulation applies to system of the form
$$\dot{z} = A_0 z + B_0 u \tag{52}$$
in order to stabilize the physical state z around some desired value z_d. We assume for the sake of simplicity that z_d is a steady state for (52) that corresponds to the steady control $u_d = 0$. Using the extended system with the extended state $x = (z^T, z_d^T)$ and the extended dynamic built up using (52) with $\dot{z}_d = 0$, one obtains the controlled system model given by:
$$\dot{x} = A_s x + B_s u \tag{53}$$
where x is an extended state containing the set-point and disturbance model state and where the cost function (36) is given by:
$$\bar{\ell}(s, p, x) := \frac{1}{2}\big[q(\bar{x}(s, p, x)) + \|\mathcal{U}(s, p)\|_R^2\big] \tag{54}$$
where $q(x)$ is given by:
$$q(x) = \|z - z_d\|_Q^2 := \|Cx\|_Q^2 \tag{55}$$
The control parametrization map $\mathcal{U}(\cdot, p)$ used in (56) gives the control profile over the prediction horizon as a function of the finite dimensional parameter vector p.

This formulation leads to state-dependent QP where the cost function and the constraints are given by:
$$f_0(p, x) = \frac{1}{2} p^T H p + (F_1 x)^T p + x^T S x \tag{56}$$
$$Ap \leq B^{(0)} + B^{(1)} x \tag{57}$$
It results that the definition of L_0, L_ψ and μ_0 remains unchanged since these parameters depends only on the state independent quantities H and A.

It is also assumed that the formulation involves appropriate final constraints such that (37) of Assumption 5.5 holds with $\Delta(\tau, x)$ satisfying:
$$\Delta(\tau, x) \geq \int_0^\tau q(\bar{x}(s, p^{opt}, x)) ds \tag{58}$$
This can be obtained through appropriate final equality constraints that can be explicitly embedded in the control parametrization map $\mathcal{U}(\cdot, p)$ or through softened final inequality constraints as suggested in [10].

Given a set of interest $\mathbb{X}$, the upper bound on D_0 defined by (23)-(25) can be used provided that $\|F\|$ is replaced by
$$\sup_{x \in \mathbb{X}} \|F_1 x\| \leq \|F_1\| \times \varrho(\mathbb{X}) \tag{59}$$
The computation of ψ^{max} invoked in (31) of proposition 5.1 is obtained according to:
$$\psi^{max} := \max_{x \in \mathbb{X}} \left[\sum_{i=1}^{n_c} \big(\max\{0, M_i x - L_i\}\big)^2\right] \tag{60}$$
where
$$\begin{aligned} M_i &:= -\Big[A_i H^{-1} F_1 + B_i^{(1)}\Big] \\ L_i &:= B_i^{(0)} \end{aligned}$$
where A_i and $B_i^{(j)}$ denote the i-th line of A and $B^{(j)}$ respectively. Note that the optimization problems (60) can be computed once for all using available NLP solvers for a

beforehand given sets of interest $\mathbb{P}$ and $\mathbb{X}$.

Once ψ^{max} is computed, the resulting κ_0^{max} involved in Proposition 5.1 [item (4)] can be computed and used in the computation of ρ^{max}. Finally, the parameter f_0^{max} involved in Proposition 5.1 is computed according to:

$$f_0^{max} := \max_{(p,x)\in\mathbb{P}\times\mathbb{X}} \left[\begin{pmatrix} p \\ x \end{pmatrix}^T \underbrace{\begin{pmatrix} H & F_1 \\ F_1^T & S \end{pmatrix}}_{:=W} \begin{pmatrix} p \\ x \end{pmatrix} \right]$$

which admits the upper bound:

$$\begin{aligned} f_0^{max} &\leq [\lambda_{max}(W)] \times \max_{z\in\mathbb{P}\times\mathbb{X}} \|z\|^2 \\ &= \lambda_{max}(W) \times \varrho(\mathbb{P}\times\mathbb{X}) \end{aligned}$$

It remains to give explicit computation of $D_{\mathbb{C}}$ in (38) of Assumption 5.6. This is given by the following proposition:

Proposition 5.3: If the constraints $p \in \mathbb{P}$ implies that $\mathcal{U}(s,p) \in \mathbb{U}$ for some compact set $\mathbb{U}$, then the following expression of $D_{\mathbb{C}}$ meets the requirement of Assumption 5.6:

$$D_{\mathbb{C}} = \lambda_{max}(Q) \times \varrho(\mathbb{X}) \times [\|A_s\|\varrho(\mathbb{X}) + \|B_s\|\varrho(\mathbb{U})] \quad (61)$$

PROOF. Compute the derivative of $\|\bar{C}x(s,p,x)\|_Q^2$ (which takes the values $q(x)$ at $s = 0$) and derive a lower bound on the speed with which this term may converge to 0 given the compact set to which belongs the arguments x and p. □

The next result concerns the explicit derivation of the compact set $\mathbb{C}_\phi$ given an initial cost function level ϕ as described in Assumption 5.3. This is the aim of the following result:

Proposition 5.4: **If** The possible set points z_d belong to a compact set $\mathbb{Z}_d$, **then** given ϕ, the compact set $\mathbb{C}_\phi$ involved in (33) of Assumption 5.3 is given by $\mathbb{C}_\phi := \mathbb{P}\times\mathbb{X}$ where:

$$\begin{aligned} \mathbb{P} &:= \Big\{p \quad \text{s.t} \quad \|p\| \leq [\phi/\lambda_{min}(W_0)]^{\frac{1}{2}}\Big\} && (62) \\ \mathbb{X} &:= \Big\{x \quad \text{s.t} \quad \|x\| \leq \varrho(\mathbb{Z}_d) + [\phi/\lambda_{min}(W_0)]^{\frac{1}{2}}\Big\} && (63) \end{aligned}$$

where W_0 is the matrix given by:

$$W_0 = \begin{pmatrix} H & F_{11} \\ F_{11}^T & S_{11} \end{pmatrix} \quad (64)$$

where $F_{11} \in \mathbb{R}^{n_p\times n_z}$ and $S_{11} \in \mathbb{R}^{n_z\times n_z}$ are the z-corresponding sub-matrices of F_1 and S involved in (56) respectively.

PROOF See Appendix J.

Once the compact set $\mathbb{C}_\phi$ is computed for a given initial value ϕ of the cost function, the constants $K_{\mathbb{C}}^0$ and $K_{\mathbb{C}}^1$ involved in (34) and (35) of Assumption 5.4 can be explicitly computed using the following proposition:

Proposition 5.5: For the cost function f_0 defined by (56) and the constraints defined by (57), given a compact set $\mathbb{C} := \mathbb{P}\times\mathbb{X}$, the constants $K_{\mathbb{C}}^0$ and $K_{\mathbb{C}}^1$ involved in (34) and (35) of Assumption 5.4 can be given by:

$$\begin{aligned} K_{\mathbb{C}}^0 &:= \|F_1^T\| \times \varrho(\mathbb{P}) + 2\lambda_{max}(S) \times \varrho(\mathbb{X}) && (65) \\ K_{\mathbb{C}}^1 &:= 2n_c\,[\psi^{max}] \cdot \|(B^{(1)})^T\| && (66) \end{aligned}$$

where ψ^{max} is computed by (60).

PROOF. See Appendix K.

The only remaining parameters are $E_{\mathbb{C}}^0$ and $E_{\mathbb{C}}^1$ involved in (28) of Assumption 5.2 and which describe the prediction error on the extended state x as a function of τ. Note that if the model is perfectly known, the only prediction error comes from the fact that the future evolution of the set-point z_d is unknown. Two cases can be distinguished:

- If the set point if filtered, then

$$E_{\mathbb{C}}^0 = 0 \text{ and } E_{\mathbb{C}}^1 = \max_t(\|\dot{z}_d(t)\|) \quad (67)$$

- Otherwise

$$E_{\mathbb{C}}^0 = \varrho(\mathbb{Z}_d) \text{ and } E_{\mathbb{C}}^1 = \max_t(\|\dot{z}_d(t)\|) \quad (68)$$

In case other sources of prediction errors prevail, then an additional positive term e_1 has to be added so that $E_{\mathbb{C}}^1 = \max_t(\|\dot{z}_d(t)\|) + e_1$ is used.

1) Illustrative example: MPC control of a chain of integrators: Let us consider MPC control of a chain of n integrators given by:

$$\begin{aligned} \dot{z}_i &= z_{i+1} \quad \text{for } i = 1,\dots,n-1 && (69) \\ \dot{z}_n &= u \quad \text{under } |u| \leq \bar{u} = 10 && (70) \end{aligned}$$

in which the objective is to track a reference trajectory on z_1 under the state constraints:

$$\begin{pmatrix} -2 \\ -1 \end{pmatrix} \leq \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} \leq \begin{pmatrix} +2 \\ +1 \end{pmatrix} \quad (71)$$

using the formulation of section V-C. This is obviously a very important sub-class of systems that is heavily used in Mechatronics.

We consider a parametrization of the form:

$$\mathcal{U}(s, p_u(t)) = [\Phi_u(s)]\, p_u(t) \qquad ; \quad p_u \in \mathbb{R}^m \quad (72)$$

in which a final constraint on the state is imposed:

$$\|z(T) - Z_d\| = Cx(T) = 0 \quad (73)$$

By doing this, the stability of the ideal perfect scheme is guaranteed with Assumption 5.5 satisfied. The final constraint satisfaction can be imposed through the reduced parametrization:

$$p_u = Kp + Mx_0 \quad (74)$$

where the matrices K and M depend on the function basis ϕ_u involved in (72) and the prediction horizon T such that taking $p = 0$ always leads to p_u that satisfies the final constraint. This means that because of the saturation constraint (70), the final

constraint (73) can be feasible through (74) only for initial state x_0 such that

$$\|Mx_0\| \leq \bar{u} \tag{75}$$

leading to the bound $\|x_0\| \leq \bar{u}/\|M\|$ ($\approx$ 11.3 in the case $n = 4$). By doing this, the number of decision variables is given by $n_p = m - n$. In the following results, the weighting matrices $Q = \mathbb{I}_n$ and $R = 0.001$ are systematically used in (55) and (56). A prediction horizon $T = 10$ is used in the sequel while $m = 10$ dimensional parametrization variable is used in control parametrization (72). This leads to a number of free decision variables of dimension $n_p = 6$.

The computation time for a single iteration $\tau_c = 0.1\mu s$ is used (time needed for a matrix-vector multiplication in Step 10 of Algorithm 2). It is supposed that the reference value z_d lies in the domain $[-5, +5]$. All the constraints are taken to be soft with $\varepsilon_\psi = 10^{-2}$. The formulation of the problem and the choice of the constraints checking instants lead to a number of constraints $n_c = 300$.

Note that in order to check the existence of q_{min} satisfying the condition (47) of Proposition (5.2), one can check the existence of solution ε_0^* to the inequality

$$R_{\tau_c}(\varepsilon_0, \varepsilon_\psi, \bar{q}_{min}) \leq -\left[\frac{\gamma_c \bar{q}_{min}^2}{3D_{\mathbb{C}_{\phi_0}}}\right] \tag{76}$$

since if (76) is satisfied for $\bar{q}_{min}$ in the l.h.s, it will be satisfied for any $\bar{q} \geq \bar{q}_{min}$ used in the l.h.s while $\bar{q}_{min}$ is used in the r.h.s. An additional condition invoked in Proposition 5.2 states that this solution ε_0^* must be such that:

$$\varepsilon_0^* \leq \left[\frac{\gamma_c \bar{q}_{min}^2}{2D_{\mathbb{C}_{\phi_0}}}\right] \tag{77}$$

Note also that the parameter ϕ_0 invoked in (76) defines an upper bound on the possible initial value of the cost $f_0(p, x)$. Therefore, in the case of hot starts, the size of ϕ_0 can define the quality of the hot start. Otherwise, one can take an upper bound pessimistic value ϕ_0 by starting from $p = 0$ and taking the upper value of $f_0(0, x)$ over the set of admissible initial state defined by (75), namely:

$$\phi_0 \leq \lambda_{max}(S)\frac{\bar{u}}{\|M\|} \tag{78}$$

Based on the knowledge of ϕ_0 given by (78), the condition (76) can be checked for different candidate values of $\bar{q}_{min}$.

Figures 5 and 6 shows the results for the cases $n = 4$ and $n = 2$ respectively. More precisely, Figure 5 shows that for the quadruple integrator system under an unknown future behavior of the set-point characterized by $E_{\mathbb{C}}^1 = 0.05$, the certification conditions (76) and (77) are satisfied with $\bar{q}_{min} = 0.36$ and $\gamma_c = 0.2$.

Figure 6 shows that the certification is possible for the double integrator system with the unknown behavior of the set-point defined by $E_{\mathbb{C}}^1 = 0.3$ provided that $\bar{q}_{min} = 0.18$ and γ_c are used in (76) and (77).

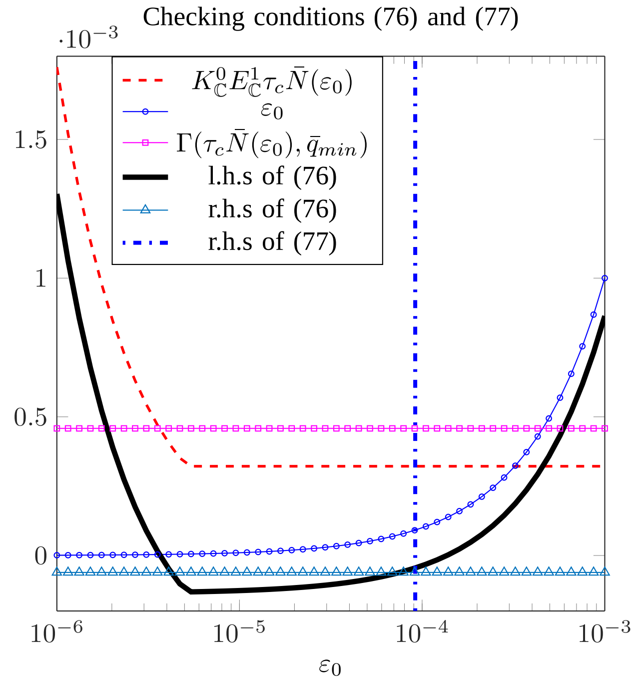

Fig. 5. Check of the certification feasibility for the chain of $n = 4$ integrators. The condition (76) is satisfied by an interval of values of ε_0 including values satisfying (77). Successful values: $\bar{q}_{min} = 0.36$, $\gamma_c = 0.2$, $E_{\mathbb{C}}^1 = 0.05$.

Once the lower bound $\bar{q}_{min}$ is computed, one can come back to the q-dependent certification condition (47) of Proposition 5.2 in order to compute for each $\bar{q} \geq \bar{q}_{min}$ the lower bound $\underline{\varepsilon}_0(\bar{q})$ and the upper bounds $\bar{\varepsilon}_0(\bar{q})$ of the admissible values of ε_0.

The state dependent sampling (48) can therefore be defined by the number of iteration associated to the precision ε_0^{sol} given by:

$$\varepsilon_0^{sol}(x) := (1 - \lambda)\underline{\varepsilon}_0(q(x)) + \lambda\bar{\varepsilon}_0(q(x)) \tag{79}$$

where $\lambda \in [0.5, 0.9]$ in order to enhance high sampling period (higher values of ε_0) while keeping some security margin.

Figures 7 shows the corresponding evolutions of the bounds $\underline{\varepsilon}_0(q)$ and $\bar{\varepsilon}_0(q)$ as functions of the ratio $q/\bar{q}_{min}$ for the quadruple integrator system ($n = 4$) studied previously. This Figure clearly shows that when $q(x)$ is high, law precision (high values of ε_0) can be used. Although this is a known fact, the results proposed here gives a certified explicit computation of this feature. The Figure shows also clearly that attempt to achieve over-precise solution may lead to instability since there is a lower bounds on ε_0.

VI. Conclusion

In this paper a certification bound on the convergence of the fast gradient algorithm when applied to solve convex

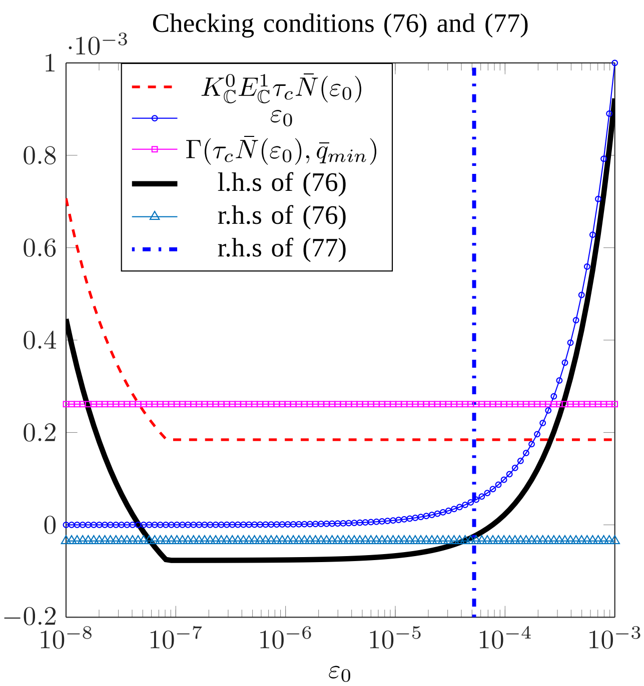

Fig. 6. Check of the certification feasibility for the chain of $n = 2$ integrators. The condition (76) is satisfied by an interval of values of ε_0 including values satisfying (77). Successful values: $\bar{q}_{min} = 0.18$, $\gamma_c = 0.2$, $E^1_{\mathbb{C}} = 0.3$.

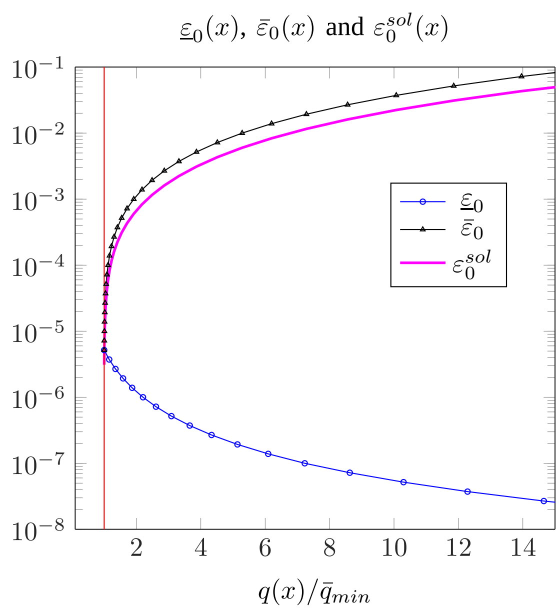

Fig. 7. Quadruple integrator: Evolution of the bounding values $\underline{\varepsilon}_0(q(x))$ and $\bar{\varepsilon}_0(q(x))$ and a possible state dependent precision $\varepsilon_0^{sol}(x)$ defined by (79).

optimization problems with general inequality constraints with a prescribed level of sub-optimality is first given. The resulting bound is then used to derive a real-time implementation of MPC with state-dependent updating period leading to certified convergence of the resulting closed-loop to a neighborhood of the desired set-point. The proposed results clearly showed that the time needed to perform the elementary iteration is a key parameter in the resulting MPC implementation. To this respect, the proposed results can be used to afford limited computational power or to compute, for a given control problem and a given specification in terms of optimality and constraints fulfillment, the admissible computation power that need to be assigned.

APPENDIX

A. Proof of Lemma 3.1

This comes from the fact that p^* is the unconstrained optimum of f which means that:

$$f_0(p^*) + \rho\psi(p^*) \le f_0(p_a) + \psi(p_a) = f_0(p_a) \tag{80}$$

and since $\psi(p^*) \ge 0$, the last inequality gives $f_0(p^*) \le f_0(p_a)$. The inequality to be proved is therefore a simple consequence of the definition 3.1 of D_0. $\square$

B. Proof of Lemma 3.2

Let us denote by p_ψ the closest element of $\mathcal{A}_{\psi=0}$ to p^*. The triangular inequality implies:

$$\|p - p_\psi\| \quad \le \quad \|p - p^*\| + \|p^* - p_\psi\| \le d(p) + \|p^* - p_\psi\| \tag{81}$$

and because $\psi \in \mathcal{F}^1_{L_\psi}$:

$$\psi(p) \le \psi(p_\psi) + \langle \psi'(p_\psi), p - p_\psi \rangle + \frac{L_\psi}{2}\|p - p_\psi\|^2 \tag{82}$$

but since $p_\psi \in \mathcal{A}_{\psi=0}$, one has that $\psi(p_\psi) = 0$ and $\psi'(p_\psi) = 0$ [because of the particular structure of the penalty], therefore (82) becomes (because of (81)):

$$\psi(p) \le \frac{L_\psi}{2}\|p - p_\psi\|^2 \le \frac{L_\psi}{2}\Big[d(p) + \|p^* - p_\psi\|\Big]^2 \tag{83}$$

It remains to prove that the term $\|p^* - p_\psi\|$ can be bounded so that the inequality (10) holds. Note that since p^* minimizes f, one has:

$$f_0(p_\psi) + \rho\psi(p_\psi) \ge f_0(p^*) + \rho\psi(p^*)$$

and since $\psi(p_\psi) = 0$ the last inequality leads to:

$$\psi(p^*) \quad \le \quad \frac{1}{\rho}\big[f_0(p_\psi) - f_0(p^*)\big] \tag{84}$$

$$\le \quad \frac{1}{\rho}\Big[\|f_0'(p^*)\| \cdot \|p^* - p_\psi\| + \frac{L_0}{2}\|p^* - p_\psi\|^2\Big] \tag{85}$$

Now let p_u be the unconstrained minimizer of f_0, namely $f_0'(p_u) = 0$. Note that p_u is uniquely defined since $\mu_0 > 0$ by assumption. Now by definition of p^* and p_u, one has:

$$f_0(p^*) + \rho\psi(p^*) \le f_0(p_u) + \rho\psi(p_u) \tag{86}$$

on the other hand,

$$f_0(p^*) \geq f_0(p_u) + \frac{\mu_0}{2}\|p^* - p_u\|^2 \tag{87}$$

By combining (86)-(87), it comes that:

$$\|p^* - p_u\| \leq \sqrt{\frac{2\rho}{\mu_0}\psi(p_u)}$$

This with the Lypschitz induced inequality gives:

$$\|f_0^{'}(p^*) - 0\| \leq L_0\|p^* - p_u\| \leq L_0\sqrt{\frac{2\rho}{\mu_0}\psi(p_u)} =: \kappa_0^{'}\sqrt{\rho}$$

where $k_0^{'} := L_0\sqrt{2\psi(p_u)/\mu_0}$.This last inequality together with (85) implies:

$$\psi(p^*) \leq \frac{1}{\rho}\big[\kappa_0^{'}\sqrt{\rho}\|p^* - p_\psi\| + \frac{L_0}{2}\|p^* - p_\psi\|^2\big] \tag{88}$$

Now using (8) in which $d(p^*, \mathcal{A}_{\psi=0}) = \|p^* - p_\psi\|$ gives:

$$\beta\|p^* - p_\psi\|^2 \leq \frac{1}{\rho}\big[\kappa_0^{'}\sqrt{\rho}\|p^* - p_\psi\| + \frac{L_0}{2}\|p^* - p_\psi\|^2\big]$$

and after straightforward manipulations, it comes that:

$$\Big[\beta - \frac{L_0}{2\rho}\Big]\|p^* - p_\psi\| \leq \frac{\kappa_0^{'}}{\sqrt{\rho}} \tag{89}$$

Now assuming that $\rho \geq L_0/\beta$, one obtains:

$$\|p^* - p_\psi\| \leq \frac{2\kappa_0^{'}}{\beta\sqrt{\rho}} = \frac{2L_0\sqrt{2\psi_0(p_u)/\mu_0}}{\beta\sqrt{\rho}}$$

which together with (83) clearly ends the proof since the inequality (11) is a direct consequence of the fact that $d(p^*) = 0$ by definition. □

C. Proof of Lemma 3.3

Since p_ψ is admissible and p^{opt} is the optimal solution of the constrained problem, one necessarily has:

$$f_0(p^{opt}) \leq f_0(p_\psi) \tag{90}$$

Moreover, since $f_0 \in \mathcal{F}^1_{L_0}$, the following inequality holds:

$$f_0(p_\psi) \leq f_0(p^*) + \|f_0^{'}(p^*)\| \cdot \|p^* - p_\psi\| + \frac{L_0}{2}\|p^* - p_\psi\|^2$$

and since $\|f_0^{'}(p^*)\| \leq D_0$ (Lemma 3.1):

$$f_0(p_\psi) \leq f_0(p^*) + D_0\|p^* - p_\psi\| + \frac{L_0}{2}\|p^* - p_\psi\|^2 \tag{91}$$

which together with (90) and (8) of Assumption 3.4 gives:

$$|f_0(p^{opt}) - f_0(p^*)| \leq D_0\left[\frac{\psi(p^*)}{\beta}\right]^{\frac{1}{2}} + \frac{L_0}{2}\left[\frac{\psi(p^*)}{\beta}\right] \tag{92}$$

This obviously ends the proof. □

D. Proof of Lemma 3.4

Assume that for some p the following inequality hold:

$$|f(p) - f(p^*)| \leq \epsilon \tag{93}$$

this means that ($f \in \mathcal{S}^1_{\mu_0}$):

$$\|p - p^*\| \leq \left[\frac{2\epsilon}{\mu_0}\right]^{\frac{1}{2}} \tag{94}$$

on the other hand:

$$|f_0(p) - f_0(p^*)| \leq D_0\|p - p^*\| + \frac{L_0}{2}\|p - p^*\|^2 \tag{95}$$

this together with (94) gives the result. □

E. Proof of Proposition 4.2

PROOF. We shall first prove that when the algorithm stops, one has:

$$|f(\hat{p}^*) - f(p^*)| \leq \eta \tag{96}$$

then we prove that when (96) holds then $\hat{p}^*$ is an $\bar{\varepsilon}$-suboptimal solution of the original problem. To prove (96), we shall distinguish two situations depending on the exit condition of step 10. Indeed, either $g(p_i) \leq g_{min} := \mu_0\sqrt{2\eta/L}$ in which case (96) is satisfied since $f \in \mathcal{S}^1_{\mu_0,L}$. Or the algorithm stops after $\bar{N}(c, \gamma_0)$ iterations where $\gamma_0 := \eta\mu_0/[(L + \mu_0)f_0(p_0)]$ which implies (96) by virtue of Corollary 4.

We shall now prove that when (96) holds, one necessarily has:

$$|f_0(\hat{p}^*) - f_0(p^{opt})| \leq \varepsilon_0 \quad ; \quad \psi(\hat{p}^*) \leq \varepsilon_\psi^2 \tag{97}$$

Proof of $\psi(\hat{p}^*) \leq \varepsilon_\psi^2$

By the μ_0-strong convexity of f, equation (96) implies that $\|\hat{p}^* - p^*\| \leq \sqrt{(2/\mu_0)\eta}$. Injecting this in (10) gives:

$$\psi(\hat{p}^*) \leq \frac{L_\psi}{2}\left[\sqrt{\frac{2\eta}{\mu_0}} + \frac{\kappa_0}{\sqrt{\rho}}\right]^2$$

So in order to prove that $\psi(\hat{p}^*) \leq \varepsilon_\psi^2$, it is sufficient to prove the following two inequalities:

$$\sqrt{\frac{2\eta}{\mu_0}} \leq \frac{\varepsilon_\psi}{2}\sqrt{\frac{2}{L_\psi}} \quad \text{and} \quad \frac{\kappa_0}{\sqrt{\rho}} \leq \frac{\varepsilon_\psi}{2}\sqrt{\frac{2}{L_\psi}}$$

But the first inequality is satisfied because $\eta \leq \eta_2$ while the second is satisfied because $\rho \geq \rho_1$.

Proof of $|f(\hat{p}^*) - f_0(p^{opt})| \leq \varepsilon_0$

Using the triangular inequality:

$$|f_0(\hat{p}^*) - f_0(p^{opt})| \leq |f_0(\hat{p}^*) - f_0(p^*)| + |f_0(p^*) - f_0(p^{opt})|$$

and using (15) and (12) the last inequality gives:

$$|f_0(\hat{p}^*) - f_0(p^{opt})| \leq D_0\left[\frac{2\eta}{\mu_0}\right]^{\frac{1}{2}} + \frac{L_0}{2}\left[\frac{2\eta}{\mu_0}\right] + D_0\left[\frac{\psi(p^*)}{\beta}\right]^{\frac{1}{2}} + \frac{L_0}{2}\left[\frac{\psi(p^*)}{\beta}\right]$$

therefore, the result can be obtained if the following inequality are satisfied:

$$D_0\left[\frac{2\eta}{\mu_0}\right]^{\frac{1}{2}}+\frac{L_0}{2}\left[\frac{2\eta}{\mu_0}\right] \quad\leq\quad \frac{\varepsilon_0}{2} \tag{98}$$

$$D_0\left[\frac{\psi(p^*)}{\beta}\right]^{\frac{1}{2}}+\frac{L_0}{2}\left[\frac{\psi(p^*)}{\beta}\right] \quad\leq\quad \frac{\varepsilon_0}{2} \tag{99}$$

The first inequality is satisfied since $\eta \leq \eta_1$ while the second is satisfied if:

$$\left[\frac{\psi(p^*)}{\beta}\right]^{\frac{1}{2}} \leq Z_1(\frac{\varepsilon_0}{2}) \tag{100}$$

But thanks to (11) [satisfied since $\rho \geq \rho_3$] this can be proved if the following inequality holds:

$$\frac{L_\psi\kappa_0^2}{2\beta\rho} \leq Z_1^2(\frac{\varepsilon_0}{2}) \tag{101}$$

which is satisfied because $\rho \geq \rho_2$. □

F. Proof of Proposition 4.3

Recall that in the specific case of QP problem, the definition of D_0 becomes

$$D_0 := \sup_{f_0(p)\leq f_0(p_a)} \|Hp+F\|$$

But we have by assumption $\|p_a\| \leq p_{max}$, which enables to write:

$$f_0(p_a) \leq \frac{1}{2}\lambda_{max}(H)\left[\varrho(\mathbb{P})\right]^2+\|F\|\cdot\varrho(\mathbb{P})+\phi_0 =: f$$

and since $f_0(p) \geq \frac{1}{2}\lambda_{min}(H)\|p\|^2-\|F\|\|p\|+\phi_0$, the last inequality implies:

$$\|p\| \quad\leq\quad \frac{\|F\|+\sqrt{\|F\|^2+2\lambda_{min}(H)\left[f-\phi_0\right]}}{\lambda_{min}(H)} =: \bar{p}$$

which obviously gives the results. □

G. Proof of Lemma 5.1

Using Assumption 5.4 and 5.2 , it comes that:

$$\begin{aligned} f_0(\hat{p}^*(t_{k+1}),x(t_{k+1})) \leq f_0(\hat{p}^*(t_{k+1}),\hat{x}(t_{k+1}))+ \\ +K_{\mathbb{C}}^0\times\left[E_{\mathbb{C}}^0+E_{\mathbb{C}}^1\times\tau_k\right] \end{aligned} \tag{102}$$

Now by definition of τ_k, the solution $\hat{p}^*(t_{k+1})$ satisfies

$$f_0(\hat{p}^*(t_{k+1}),\hat{x}(t_{k+1})) \leq f_0(p^{opt}(t_{k+1}),\hat{x}(t_{k+1}))+\varepsilon_0^{(k+1)}$$

which together with Assumption 5.5 gives:

$$\begin{aligned} f_0(\hat{p}^*(t_{k+1}),\hat{x}(t_{k+1})) \leq \quad & f_0(p^{opt}(t_k),x(t_k))+\varepsilon_0^{(k+1)} \\ & -\Delta(\tau_k,x(t_k)) \\ \leq \quad & f_0(\hat{p}^*(t_k),x(t_k))+\varepsilon_0^{(k)} \\ & +\varepsilon_0^{(k+1)}-\Delta(\tau_k,x(t_k)) \end{aligned} \tag{103}$$

Using the last inequality in (102) gives the result. □

H. Proof of Lemma 5.2

By definition of (37) of Δ and using (38) of Assumption (5.6), it comes that:

$$\begin{aligned} \Delta(\tau,x) \quad&\geq\quad \int_0^\tau \max\{0,q(x)-D_{\mathbb{C}}s\}ds \\ &=\quad \int_0^{\min\{\tau,q(x)/D_{\mathbb{C}}\}}(q(x)-D_{\mathbb{C}}s)ds \\ &=\quad \left[q(x)\tau-\frac{1}{2}D_{\mathbb{C}}\tau^2\right]_0^{\min\{\tau,q(x)/D_{\mathbb{C}}} \end{aligned}$$

which can be expressed using $\Gamma_{\mathbb{C}}(\tau,q)$ given by (44). □

I. Proof of Proposition 5.2

The first inequality in (50) together with Assumption 5.3 impliy that Corollary 5 applies with $k=0$, $\mathbb{C}=\mathbb{C}_{\phi_0}$ and $\bar{q} := q(x(t_k))$, therefore one has:

$$\begin{aligned} & f_0(\hat{p}^*(t_1),x(t_1))-f_0(\hat{p}^*(t_0),x(t_0)) \leq \\ & \varepsilon_0^{(0)}+R_{\tau_c}(\varepsilon_0^{(1)},\varepsilon_\psi,q(x(t_0))) \end{aligned} \tag{104}$$

and since $\varepsilon_0^{(1)}=\varepsilon_0^{sol}(x(t_0))$, if $q(x(t_0))>\bar{q}_{min}$ the inequality (47) gives:

$$R_{\tau_c}(\varepsilon_0^{(1)},\varepsilon_\psi,q(x(t_0))) \leq -\frac{\gamma_c\bar{q}_{min}^2}{3D_{\mathbb{C}_{\phi_0}}} \tag{105}$$

and thanks to the second inequality in (50), the inequality (105) gives:

$$\varepsilon_0^{(0)}+R_{\tau_c}(\varepsilon_0^{(1)},\varepsilon_\psi,q(x(t_0))) \leq -\frac{\gamma_c\bar{q}_{min}^2}{6D_{\mathbb{C}_{\phi_0}}} \tag{106}$$

This together with (104) implies that $f_0(\hat{p}^*(t_1),x(t_1))$ decreases meaning that the new pair is still in $\mathbb{C}_{\phi_0}$ and since $\varepsilon_0^{(1)}$ satisfies by assumption the second inequality in (50), the argumentation can be repeated to derive the properties of the next pair $(\hat{p}^*(t_2),x(t_2))$ meaning that the following inequality:

$$f_0(\hat{p}^*(t_{k+1}),x(t_{k+1}))-f_0(\hat{p}^*(t_k),x(t_k)) \leq -\frac{\gamma_c\bar{q}_{min}^2}{6D_{\mathbb{C}_{\phi_0}}}$$

is satisified as far as $q(x(t_k))$ remains grater than $\bar{q}_{min}$. This clearly implies that $x(t_k)$ converges to the limit set $\mathbb{X}_{min}$ defined by (49).

regarding the constraints, note that the hard constraints are necessarily satisfied since they depend only on p by assumption and that $\hat{p}^*(t_{k+1})$ satisfies by construction the hard constraints while allowing only for a violation of the soft constraints by an amount which is lower than ε_ψ, therefore, one has:

$$c_i(\hat{p}^*(t_{k+1}),\hat{x}(t_{k+1})) \leq \varepsilon_\psi \quad \forall i\in I_s \tag{107}$$

which obviously gives (51) by Assumptions 5.2 and 5.4. □

J. Proof of Proposition 5.4

Given z_d, one rewrite the cost function using the change of variable $y = z - z_d$ enables to write the cost function (56) in the form

$$f_0(p, y) = \frac{1}{2}\begin{pmatrix} p \\ y \end{pmatrix}^T \underbrace{\begin{pmatrix} H & F_{11} \\ F_{11}^T & S_{11} \end{pmatrix}}_{W_0} \begin{pmatrix} p \\ y \end{pmatrix}$$

which means that if $f_0(p, x) \le \phi$ then the following inequalities hold:

$$\|p\| \le [\phi/\lambda_{min}(W_0)]^{\frac{1}{2}} \ ; \ \|y\| \le [\phi/\lambda_{min}(W_0)]^{\frac{1}{2}} \tag{108}$$

The first inequality obviously gives (62) while the second leads to:

$$\|z\| \le \|z_d\| + [\phi/\lambda_{min}(W_0)]^{\frac{1}{2}}$$

which gives (63). $\square$

K. Proof of Proposition 5.5

In order to prove that (65) satisfies (34), we use the definition of f_0 to write:

$$\begin{aligned} &\|f_0(p, x_1) - f_0(p, x_2)\| = \\ &\left\|(F_1(x_1 - x_2))^T p + \|x_1\|_S^2 - \|x_2\|_S^2\right\| \\ &\le \|(F_1^T p)^T (x_1 - x_2)\| + 2\lambda_{max}(S) \times \varrho(\mathbb{X}) \times \|x_1 - x_2\| \\ &\le \left[\|F_1^T\| \times \varrho(\mathbb{P}) + 2\lambda_{max}(S) \times \varrho(\mathbb{X})\right] \cdot \|x_1 - x_2\| \end{aligned}$$

which proves (65).

It remains to prove that $K_{\mathbb{C}}^1$ defined by (66) satisfies (35) we first note that:

$$\psi(x) := \sum_{i=1}^{n_c} [r_i(p, x)]^2$$

with $r_i(p, x) = \max\{0, A_i p - B_i^0 - B_i^1 x\}$. Therefore:

$$\|\frac{\partial \psi}{\partial x}\| \le 2\sum_{i=1}^{n_c} |r_i(p, x)| \cdot \|B_i^{(1)}\|$$

and using the inequalities expressing the equivalence of the L_2 and L_1 norms, the last inequality gives:

$$\begin{aligned} \|\frac{\partial \psi}{\partial x}\| \quad &\le \quad 2n_c \sum_{i=1}^{n_c} |r_i(p, x)|^2 \cdot \|(B^{(1)})^T\| \\ &\le \quad 2n_c \left[\psi(p, x)\right] \times \|(B^{(1)})^T\| \\ &\le \quad 2n_c \times \psi^{max} \times \|(B^{(1))^T}\| \end{aligned}$$

which obviously gives (66). $\square$

REFERENCES

[1] M. Alamir. Optimization based nonlinear observers revisited. *International Journal of Control*, 72(13):1204–1217, 1999.

[2] M. Alamir. Monitoring control updating period in fast gradient-based NMPC. In *Proceedings of the European Control Conference (ECC2013)*, Zurich, Switzerland, 2013.

[3] M. Alamir. Fast MPC, a reality steered paradigm: Key properties of fast NMPC algorithms. In *Proceedings of the European Control Conference (ECC2014)*, Strasbourg, France, 2014.

[4] L. T. Biegler. *Nonlinear Programming*. Society for Industrial and Applied Mathematics, 2010.

[5] A. Bomporad and P. Patrinos. Simple and certifiable quadratic programming algorithms for embedded linear model predictive control. In *Proceeding of the IFAC Nonlinear Predictive Control Conference*, Noordwijkerhout, NL, 2012.

[6] F. Bonne, M. Alamir, and P. Bonnay. Experimental investigation of control updating period monitoring in industrial PLC-based fast MPC: application to the constrained control of a cryogenic refrigerator. arxiv:1406.6281. 2012.

[7] A. Domahidi, A. U. Zgraggen, M. N. Zeilinger, and M. Morariand C. Jones. Efficient interior point methods for multistage problems arising in receding horizon control. In *Proceeding of the IEEE Conference on Decision and Control*, Hawaii, USA, 2012.

[8] H. J. Ferreau, H. G. Bock, and M. Diehl. An on line active set strategy to overcome the limitations of explicit mpc. *Int. J. Robust Nonlinear Control*, 18:816–830, 2008.

[9] C. N. Jones, A. Domahidi, M. Morari, S. Richter, F. Ullmann, and M. Zeilinger. Fast predictive control: Real-time computation and certification. In *Proceeding of the IFAC Nonlinear Predictive Control Conference*, Noordwijkerhout, NL, 2012.

[10] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. Scokaert. Constrained model predictive control: Stability and optimality. 36:789–814, 2000.

[11] L. K. McGovern. Computationak abalysis of real-time convex optimization for control systems. Ph.D. Dissertation, Massachusetts Institute of Technology. Department of Aeronautics and Astronautics, Cambridge, MA, USA. 2000.

[12] Y. Nesterov. A method of solving a convex programming problem with convergence rate O(1/k^2). *Soviet Mathematics Doklady*, 27(2):372–376, 1983.

[13] Y. Nesterov. *Introductory lectures in convex optimization: a basic course*. Kluwer Academic Publishers, 2004.

[14] S. Richter, C.N. Jones, and M. Morari. Computational complexity certification for real-time mpc with input constraints based on the fast gradient method. *Automatic Control, IEEE Transactions on*, 57(6):1391–1403, June 2012.

[15] V. M. Zavala, C. D. Laird, and L. T. Biegler. Interior point decomposition approaches for parallel solution of large scale nonlinear parameter estimation problems. *Chemical Engineering Science*, 63(19):4834–4845, 2008.

[16] Zhiwen Zhu and H. Leung. Adaptive identification of nonlinear systems with application to chaotic communications. *Circuits and Systems I: Fundamental Theory and Applications, IEEE Transactions on*, 47(7):1072–1080, Jul 2000.